Math 6261 2023-03-17

Review.

Thm (CLT) Let (X_n) be an i.i.d. sequence of r.v.'s with

$$E(X_n)=0 \quad \text{and} \quad Var(X_n)=\sigma^2.$$

Write $S_n = X_1+\dots+X_n$. Then for any $x\in\mathbb{R}$,

$$P\left(\frac{S_n}{\sqrt{n}\,\sigma}\le x\right) \longrightarrow \Phi(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy$$

as $n\to\infty$.

(standard normal distribution)

Below we present a generalized version of the CLT, which can be proved by using a similar argument.

Thm (The Lindeberg-Feller Thm)

Suppose for each n, the r.v.'s

$$X_{n,1},\ X_{n,2},\ \dots,\ X_{n,r_n}$$

are independent, with

$$E(X_{n,k})=0,\quad \sigma_{n,k}^2 = E(X_{n,k}^2) \quad \text{and} \quad s_n^2 = \sum_{k=1}^{r_n}\sigma_{n,k}^2.$$

Suppose for each $\varepsilon>0$

$$\lim_{n\to\infty}\sum_{k=1}^{r_n}\frac{1}{s_n^2}\int_{|X_{n,k}|>\varepsilon s_n} X_{n,k}^2\,dP = 0$$

Then

$$\frac{X_{n,1}+\cdots+X_{n,r_n}}{s_n} \xrightarrow{w} Z,$$

where Z has the standard normal distribution.

Corollary. Let $X_1, X_2, \cdots,$ be independent r.v.'s. Suppose that

$$E(X_n)=0 \quad \text{and} \quad |X_n| \le c \text{ uniformly.}$$

If $s_n^2 = \sum_{k=1}^{n} Var(X_n) \to \infty$ as $n\to\infty$, then

Then

$$\frac{X_1+\cdots+X_n}{s_n} \xrightarrow{w} Z,$$

where Z has the standard normal distribution.

§ 4. Random walks.

§ 4.1 Sums of independent r.v.'s.

Let $X_1, X_2, \cdots,$ be independent of r.v.'s. Let

$$S_n = X_1+\cdots+X_n.$$

We first study whether S_n converges.

Define $A = \left\{ \omega : \sum_{n=1}^{\infty} X_n(\omega) \text{ converges} \right\}$.

Notice that for each m, the values of $X_1(\omega), \cdots, X_{m-1}(\omega)$ are not relevant to the question of whether $\omega \in A$. More precisely, for given $m \in \mathbb{N}$,

$$A = \left\{ \omega : \quad \sum_{k=m}^{\infty} X_k(\omega) \text{ converges} \right\}.$$

Notice that

$\omega \in A \Leftrightarrow \forall\, \ell \in \mathbb{N}, \ \exists \ N \geq m$ such that for each $p \geq 0$

$$|X_N(\omega) + \cdots + X_{N+p}(\omega)| < \frac{1}{\ell}.$$

Hence

$$A = \bigcap_{\ell=1}^{\infty} \bigcup_{N=m}^{\infty} \bigcap_{p=0}^{\infty} \left\{ \omega : \ |X_N(\omega) + \cdots + X_{N+p}(\omega)| < \frac{1}{\ell} \right\}$$

It follows that

$$A \in \sigma(X_m, X_{m+1}, \cdots).$$

Hence $\quad A \in \bigcap_{m=1}^{\infty} \sigma(X_m, X_{m+1}, \cdots) =: \mathcal{T}.$

We call $\mathcal{T}$ the tail σ-field associated with $X_1, X_2, \cdots$.

Thm 4.1 (Kolmogrov's Zero-One law).

Suppose that $X_1, X_2, \cdots,$ are independent and that

$$A \in \mathcal{T} = \bigcap_{n=1}^{\infty} \sigma(X_n, X_{n+1}, \cdots).$$

Then either $P(A)=0$ or $P(A)=1$.

Pf. We will show that A is independent of itself. Hence

$$P(A\cap A)=P(A)P(A), \text{ i.e. } P(A)=P(A)^2,$$

and hence $P(A)=0$ or 1.

We prove this in two steps.

<u>Step 1</u>: If $B\in\sigma(X_1,X_2,\cdots,X_k)$ and $C\in\sigma(X_{k+1},X_{k+2},\cdots)$, then B and C are independent.

Clearly, $\sigma(X_1,X_2,\cdots,X_k)$ and $\bigcup_{j=1}^{\infty}\sigma(X_{k+1},\cdots,X_{k+j})$ are independent. Since the second one is a π-system, so

$$\sigma(X_1,X_2,\cdots,X_k) \text{ and } \sigma\Big(\bigcup_{j=1}^{\infty}\sigma(X_{k+1},\cdots,X_{k+j})\Big)=\sigma(X_{k+1},\cdots)$$

are independent.

<u>Step 2</u>: If $B\in\sigma(X_1,X_2,\cdots)$ and $C\in\mathcal{T}$, then B and C are independent.

For any k, since $\mathcal{T}\subset\sigma(X_{k+1},X_{k+2},\cdots)$ so $\mathcal{T}$ and $\sigma(X_1,\cdots,X_k)$ are independent. It follows that $\mathcal{T}$ and $\bigcup_{k=1}^{\infty}\sigma(X_1,\cdots,X_k)$ are independent.

Therefore $\mathcal{T}$ and $\sigma\Big(\bigcup_{k=1}^{\infty}\sigma(X_1,\cdots,X_k)\Big)=\sigma(X_1,X_2,\cdots)$ are independent.

Let $A \in \mathcal{T}$. Since $\mathcal{T} \subset \sigma(X_1, \cdots, X_k, \cdots)$, by Step 2, A is independent of itself. $\blacksquare$

By the Kolmogrov's zero-one law, if $X_1, X_2, \cdots$ are independent, then the set where $\sum_{n=1}^{\infty} X_n(\omega)$ converges has probability either 0 or 1.

Below we discuss how to determine which of 0 and 1 is the prob. of the set. First we prove two maximal inequalities.

Prop 4.2. Suppose $X_1, X_2, \cdots, X_n$ are independent r.v.'s with mean 0 and finite variance. For $\alpha > 0$,

$$P\left(\max_{1 \le k \le n} |S_k| \ge \alpha \right) \le \frac{1}{\alpha^2} \operatorname{Var}(S_n).$$

Remark: By Chebyshev inequality, $P(|S_n| \ge \alpha) \le \frac{1}{\alpha^2} \operatorname{Var}(S_n)$.

Pf of Prop 4.2.

Let $A_k = \{ \omega : |S_k(\omega)| \ge \alpha$ and $|S_j(\omega)| < \alpha$ for $j < k \}$, $k = 1, 2, \cdots$

Then A_k are disjoint.

Hence

$$E(S_n^2) \geq \sum_{k=1}^{n} \int_{A_k} S_n^2 \, dP$$

$$= \sum_{k=1}^{n} \int_{A_k} S_k^2 + 2S_k(S_n - S_k) + (S_n - S_k)^2 \, dP$$

$$\geq \sum_{k=1}^{n} \int_{A_k} S_k^2 + 2S_k(S_n - S_k) \, dP$$

Since A_k and S_k are in $\sigma(X_1, \cdots, X_k)$ and $S_n - S_k$ in $\sigma(X_{k+1}, \cdots, X_n)$,

$$\int_{A_k} 2S_k(S_n - S_k) \, dP = 2\int \mathbb{1}_{A_k} S_k (S_n - S_k) \, dP$$

$$= 2\left(\int \mathbb{1}_{A_k} S_k \, dP\right)\underbrace{\left(\int S_n - S_k \, dP\right)}_{=0}$$

$$= 0.$$

Hence

$$E(S_n^2) \geq \sum_{k=1}^{n} \int_{A_k} S_k^2 \, dP$$

$$\geq \alpha^2 \sum_{k=1}^{n} P(A_k)$$

$$= \alpha^2 \cdot P\left(\max_{1 \leq k \leq n} |S_k| \geq \alpha\right).$$

$\blacksquare$

Prop 4.3. Let $X_1, \cdots, X_n$ be independent. Then for $\alpha > 0$,

$$P\left(\max_{1 \leq k \leq n} |S_k| > 3\alpha\right) \leq 3 \max_{1 \leq k \leq n} P(|S_k| \geq \alpha).$$

Pf. Let $B_k = \{\omega : |S_k(\omega)| > 3\alpha \text{ and } |S_j(\omega)| < 3\alpha \text{ for } j < k\}$

Then $B_1, \cdots, B_n$ are disjoint with $\bigcup_{k=1}^{n} B_k = \left(\max_{1 \leq k \leq n} |S_k| > 3\alpha\right)$.

$$P\left(\max_{1 \le k \le n} |S_k| > 3\alpha \right)$$

$$\le P(|S_n| \ge \alpha) + \sum_{k=1}^{n-1} P\left(B_k \cap \{|S_n| < \alpha\}\right)$$

$$\le P(|S_n| \ge \alpha) + \sum_{k=1}^{n-1} P\left(B_k \cap \{|S_n - S_k| > 2\alpha\}\right)$$

(since B_k, $S_n - S_k$ are independent)

$$= P(|S_n| \ge \alpha) + \sum_{k=1}^{n-1} P(B_k)\, P\left(|S_n - S_k| > 2\alpha\right)$$

$$\le P(|S_n| \ge \alpha) + \max_{1 \le k \le n-1} P\left(|S_n - S_k| \ge 2\alpha\right)$$

$$\le P(|S_n| \ge \alpha) + \max_{1 \le k \le n-1} \left(P(|S_n| \ge \alpha) + P(|S_k| \ge \alpha) \right)$$

$$\le 3 \max_{1 \le k \le n} P(|S_k| \ge \alpha).$$

$\blacksquare$

Now we are ready to prove

Thm 4.4. Suppose that $X_1, X_2, \cdots,$ are independent with mean 0.

If $\sum_{n=1}^{\infty} \mathrm{Var}(X_n) < \infty$, then $\sum_{n=1}^{\infty} X_n$ converges with prob. 1.

Pf. By Prop 4.2.

$$P\left(\max_{1\le k\le r} |S_{n+k} - S_n| > \varepsilon\right)$$

$$\le \frac{1}{\varepsilon^2} \operatorname{Var}(S_{n+r} - S_n)$$

$$= \frac{1}{\varepsilon^2} \sum_{k=1}^{r} \operatorname{Var}(X_{n+k}).$$

$$\le \frac{1}{\varepsilon^2} \sum_{k=1}^{\infty} \operatorname{Var}(X_{n+k}).$$

Letting $r \to \infty$ gives

$$P\left(\sup_{k\ge 1} |S_{n+k} - S_n| > \varepsilon\right) \le \frac{1}{\varepsilon^2} \sum_{k=1}^{\infty} \operatorname{Var}(X_{n+k})$$

Hence

$$\lim_{n\to\infty} P\left(\sup_{k\ge 1} |S_{n+k} - S_n| > \varepsilon\right) = 0. \qquad (*)$$

Let $E(n,\varepsilon)$ denote the set where $\sup_{k,j\ge n} |S_k - S_j| > 2\varepsilon$

and $E(\varepsilon) = \bigcap_{n=1}^{\infty} E(n,\varepsilon)$.

Then $E(n,\varepsilon) \downarrow E(\varepsilon)$.

Since $\left(\sup_{k\ge 1} |S_{n+k} - S_n| > \varepsilon\right) \supset E(n,\varepsilon) \supset E(\varepsilon)$,

by $(*)$, $P(E(\varepsilon)) = 0$.

Notice that $\left(S_n \text{ diverges} \right) \subset \bigcup_{\varepsilon \in \mathbb{Q}_+} E(\varepsilon)$,

So $P\left(S_n \text{ diverges} \right) = 0$. $\boxtimes$

Example 4.5. Consider the random series $\sum_{n=1}^{\infty} \pm \frac{1}{n}$, where the signs are chosen on the toss of a coin. Then the series converges with prob. 1.